Almost-Fuchsian representations in $SO(4, 1)$

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Plan

1. Global Setup
2. $G = SO(3,1)$
3. $SO(4,1)$ and disk bundles

The equivariant minimal disk problem

- G rank 1, $\kappa_X < 0$, $\kappa_X \leq -1$

  $G = SO_0(n,1)$, $X = \mathbb{H}^n$.

X, G-invariant metric

G Lie group, semi-simple of noncompact type.
$X = G/K$ symmetric space. $\rho : \Gamma = \pi_1\Sigma \to G$ faithful, discrete. Is there $f : \widetilde{\Sigma} \hookrightarrow X$ minimal and equivariant ?

$\mathrm{Tr}\, \mathrm{II}_f = 0$ $\iff$ $(\mathrm{II}_f)^{(1,1)}_{\mathbb{C}} = 0$

$f(\gamma \cdot x) = \rho(\gamma) f(x)$

The Existence problem

Local Minimum

'82 '83 '83

Sacks–Uhlenbeck, Meeks–Yau, Anderson

For ρ convex-cocompact, there is an equivariant stable minimal disk

embedded [Freedman Hass Scott '83]

$\rho : \pi_1\Sigma \to SO(3,1) \simeq PSL(2,\mathbb{C})$

- def: ρ CC if $\exists$ C | convex, ρ-invariant, nonempty, $\rho \backslash C$ compact, $C \subset X$

- ρ CC $\Rightarrow \Lambda_\rho$ quasi-circle.

Uniqueness ?

'12 '15

Wang, Huang–Wang

For any $N > 0$ there are convex-cocompact representations with at least N stable equivariant minimal disks.

Q: For a fixed genus g, can $\pi_1\Sigma_g \xrightarrow{cc} PSL(2,\mathbb{C})$ admit $\geq N$ equi. min disks?

Almost-Fuchsian representations

'83

Uhlenbeck

ρ is Almost-fuchsian if there is f equivariant minimal disk proper immersion with $\underbrace{|\mathrm{II}_f| < 1}$

$\rightarrowtail$ G, $K_X \leq -1$, $|\mathrm{II}_f(u,u)| < (1-\varepsilon)|u|^2$ for $\varepsilon > 0$

- Krasnov-Schlenker '07 : almost-fuchsian immersion

AF reps 2

$PSL(2,\mathbb{C})$ $\quad\quad$ '21 $SO_0(n,1)$

Uhlenbeck, Krasnov–Schlenker, Jiang

Let ρ almost-fuchsian, then:

i) ρ is convex-cocompact ✓

ii) f is the unique equivariant minimal disk, is embedded

iii) the exponential map is a diffeomorphism $\mathbb{D}^2 \times \mathbb{R}^{n-2} \xrightarrow{\sim} \mathbb{H}^n$.

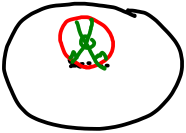
↳ $n=3$, $\rho\backslash\mathbb{H}^3 \simeq \Sigma \times \mathbb{R}$

↳ $n\geq 3$, $\bar{f} : \Sigma \to \rho\backslash\mathbb{H}^n$ $\quad$ $\exp_{\bar{f}} : N\Sigma \simeq \rho\backslash\mathbb{H}^n$

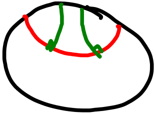
Sketch of proof

'Taylor order 2 approx. of f'

- $|\mathrm{II}_f| > 1$



- $|\mathrm{II}_f| < 1$



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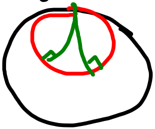
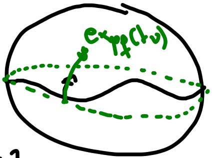


- $\sup |\mathrm{II}_f| < 1$

(...) $\exp_f^* g_{hyp}$ nondegenerate + properness

$\rightsquigarrow$ global diffeomorphism



$\rightsquigarrow$ $\exp_f(\mathbb{D}^2 \times S^{n-3})$ is mean-convex.

Holomorphic parametrization

$\mathcal{AF}_c$ injective

$\rho \longrightarrow (\rho, f) \longrightarrow (\rho, I_f, II_f) \longrightarrow (II_f)^{(2,0)}_{\mathbb{C}} \in T^*\mathrm{Teich}(\Sigma)$

$[I_f]$

F

'14   '23

Weintraub, Bronstein–Smith

The parametrization of the almost-fuchsian locus by Hopf differentials of the unique minimal disk is injective with image a fiberwise convex set of $T^*\mathrm{Teich}(S)$.

Only true in $PSL(2,\mathbb{C})$

it gives $\mathcal{AF}_c/_{\mathcal{M}}$ Mapping Class Group invariant.

$\downarrow$

$\mathrm{Teich}(\Sigma)$

$\cdot\ [\{\varphi \in H^0(K^2) \ |\varphi|_\infty \leq \tfrac{1}{2}\} \subseteq F(\mathcal{AF}_c) \subseteq \{\varphi : |\varphi|_\infty \leq 1\}]$

In codimension 2

$G = SO(4,1)$ $X = \mathbb{H}^4$

Schoen–Yau '79

For ρ convex-cocompact, there is a ~~branched~~ minimal immersion
almost-fuchsian

Questions:

- Which disk bundles can be uniformised by almost-fuchsian representations ? $N_g\Sigma \approx$ e(4,1)'s
- Can we have a holomorphic parametrization of the Almost-Fuchsian moduli space ?

Disk bundles on a Surface

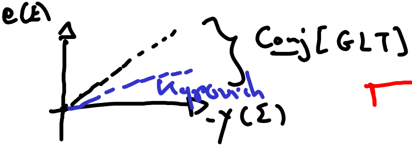
$$\begin{tikzcd} \mathbb{D}^2 \arrow[r] & E \arrow[d] \\ & \Sigma \end{tikzcd}$$

Topological classification: by the degree/ Euler class $\overset{\chi(\Sigma)}{}$. $e(E)$

Gromov–Lawson–Thurston '88

There are examples of nontrivial disk bundles over a surface admitting a convex-cocompact uniformization

Kapovich '91: if $0 < e(E) < \frac{-\chi(\Sigma)}{22}$, there is a hyp. structure on E.



Almost-Fuchsian structure

With Almost-Fuchsian ?

B. '23

There is a genus $g_0 \geq 2$ such that for any $g \geq 2$, there is a representation $\rho : \pi_1 \Sigma_g \to SO_0(4,1)$ satisfying:

i) $\rho \backslash \mathbb{H}^4$ is a degree 1 disk bundle over Σ_g.

ii) ρ is almost-fuchsian.

iii) f the equivariant minimal disk is superminimal $(\mathrm{II})^{2,0}_f \in H^0(K^2 N)$

Superminimality

Kommerell 1897.

$f : \Sigma \hookrightarrow M^4$

$(N_f\Sigma)_{\mathbb{C}} \simeq N \oplus N^{-1}$

$(\mathrm{II}_f)^{(2,0)}_{\mathbb{C}} \in \Gamma(K^2(N \oplus N^{-1}))$

$\overset{\shortparallel}{\alpha \oplus \beta} \qquad \alpha \in \Gamma(K^2N)$

$\beta \in \Gamma(K^2N^{-1})$

Superminimality: $\underbrace{\alpha \cdot \beta}_{\in \Gamma(K^4)} = 0$

$\underline{\text{in } \mathbb{H}^4}$: $\iff \begin{cases} \deg N = 0, & \alpha = \beta = 0 \\ \deg N > 0, & \beta = 0 \end{cases}$

$\underline{\text{Loftin-McIntosh}}$:

equiv. min. disk superminimal

$\Updownarrow$

β is fixed by $\mathbb{C}^\times \curvearrowright \mathcal{X}(\pi_1\Sigma, SO(4,1))$

Sketch of proof

$e^{2u} \sim$ induced metric

$e^{2v}h$ —— on N

$\Rightarrow F_{h,N} = \frac{\deg N}{2g-2}\cdot \omega$

Codazzi: $\bar\partial \alpha = 0$, $\alpha \in H^0(K^2N)$

Gauss: $\Delta u = e^{2u} - 1 + e^{-2u}e^{2v}|\alpha|^2 + e^{2w}|\beta|^2$ (E_G)

Mainardi: $\Delta v = \frac{\deg N}{2g-2} - e^{-2u}e^{2v}|\alpha|^2$ (E_M)

$\Delta w =$ ——— $- e^{-2u}e^{2w}|\beta|^2$

i) Write the curvature equations

ii) Build 2 maps such that the composition has a fixed point

iii) 2 equations –> 2 maps

(E_G): $v \mapsto \Phi(v) =$ a sol. of (E_G)

(E_M): $u \mapsto \Psi(u) = v$ —— (E_M)

· Almost-fuchsian condition: $\sup\{e^{-4u}e^{2v}|\alpha|^2\} < 1$

The Gauss' equation

$$\Delta u = e^{2u} - 1 + e^{-2u} e^{2v} |\alpha|^2 \tag{1}$$

Use Sub-Supersolution method. To get $v \mapsto \Phi(v)$ solution.

$\leadsto$ boils down to the $PSL(2,\mathbb{C})$ case.

$\mathcal{X} = \{ v : \Sigma \to \mathbb{R} \ ; \ e^{2v}|\alpha|^2 \leq \frac{1}{4} \}$. $\leadsto$ a unique $u = \Phi(v)$

$e^{2u} \in [1, 2]$.

The Mainardi's equation

$$\Delta w = 1 - e^{2w} f$$

$$\Delta v = \frac{\deg N}{2g-2} - e^{-2u} e^{2v} |\alpha|^2 \tag{2}$$

Use Moser-Trudinger inequality to get $u \mapsto \Psi(u)$

Try to maximize
$$J(w) = -\frac{2g-2}{\deg N \operatorname{Vol}\Sigma} \int |\nabla w|^2 + \log\left(\frac{1}{\operatorname{Vol}\Sigma} \int e^{2w} e^{-2u} |\alpha|^2\right)$$

The Moser-Trudinger Inequality

$$e^{2u} \lesssim e^{4\pi \frac{u^2}{|\nabla u|_2^2}} e^{\frac{|\nabla u|_2^2}{4\pi}}$$

spectral gap
systole

$$\int |\nabla u|^2 \leq 1 \quad \Rightarrow \quad \int e^{4\pi u^2} - 1 \leq C(X) \tag{3}$$

Use to show $J(w) = -\frac{2g-2}{\mathrm{Vol}\Sigma \deg N} \int |\nabla w|^2 + \log \frac{1}{\mathrm{Vol}\Sigma} \int e^{2w} e^{-2u} |\alpha|^2$ is upper bounded. $\rightsquigarrow$ works iff $\underline{\deg N = 1}$.

$\rightsquigarrow$ Ψ well-defined

Control on the solution

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if $v \in \mathcal{X}$

$\psi \circ \Phi(v) \in \mathcal{X}$

$\Updownarrow$

$C(1,2)$

To get our fixed-point arguing: we need $|e^{2u}e^{2\Psi(u)}|\alpha|^2|_\infty \leq \frac{1}{4}$.

Moser-Trudinger: $e^{2\psi(u)}|\alpha|^2 \lesssim \frac{C(\Sigma,\alpha)}{2g-2}$

. In some cases $\boxed{\frac{C(\Sigma,\alpha)}{2g-2} \leq \frac{1}{4}}$ $\rightsquigarrow$ existence $\square$.

depends $\frac{|\alpha|^2_\infty}{\int |\alpha|^2}$. One way is to take $\alpha = s \cdot f$ $\quad \boxed{s \in H^0(K^2)}$ $\quad f \in H^0(N)$

In the large genus limit