Riemannian Metrics and Finsler metrics on higher Teichmüller spaces

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Plan

1. Metrics on classical Teichmuller space

2. Higher Teichmuller spaces

3. Metrics on higher Teich spaces (and Thermodynamic formalism).

1. Metrics on classical Teichmuller space

Let S be a closed connected oriented surface genus $g \geq 2$.

The Teichmüller space $\mathcal{T}(S) \subseteq \mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{R}))/\sim_{conj}$ is a connected component consisting of discrete, faithful representations

Rmk 1) $\mathcal{T}(S) = \{$ hyperbolic metrics σ on $S\}/\sim$

$\sigma_1 \sim \sigma_2$ if $\exists\, \varphi$ diffeo isotopic to id $\varphi^* \sigma_1 = \sigma_2$

2) $\mathcal{T}(S) \underset{\text{homeo}}{\simeq} \mathbb{R}^{6g-6}$

Two interesting metrics on $\mathcal{T}(S)$:

a) Thurston's Riemannian metric

Consider $\{\gamma_n\}$ closed geodesics $X_{\rho_0} \in \mathcal{T}(S)$

becomes equidistributed:

$$\frac{\delta_{\gamma_n}}{l_{\rho_0}(\gamma_n)} \xrightarrow[\text{weak*}]{} m^L_{\rho_0} = \text{Liouville measure}$$

Define the intersection number $I(\rho_0, \rho_1)$ $\rho_0, \rho_1 \in T(S)$

$$I(\rho_0, \rho_1) := \lim_{n\to\infty} \frac{l_{\rho_1}(\gamma_n)}{l_{\rho_0}(\gamma_n)}$$

Later give another interpretation

Thurston's Riemannian metric: for $\{\rho_t\} \subseteq T(S)$

$$\langle \cdot, \cdot \rangle_P : T_{\rho_0}T(S) \times T_{\rho_0}T(S) \longrightarrow \mathbb{R}_{\geq 0}$$

$$\| \partial_t \rho_t|_{t=0} \|^2_P (\rho_0) := \partial_t^2 I(\rho_0, \rho_t) \big|_{t=0}$$

Rmk Thm (Wolpert) Thurston's Riemannian metric

= Weil Petersson metric

b) Thurston's Finsler metric (distance function)

$$d_{Th}(\cdot,\cdot): \quad T(S) \times T(S) \longrightarrow \mathbb{R}_{\geq 0}$$

$$d_{Th}(\rho_0, \rho_1) = \log \sup_{\gamma \in \pi_1(S)} \frac{l_{\rho_1}(\gamma)}{l_{\rho_0}(\gamma)}$$

is a Finsler metric on $T(S)$.

Rmk

1). One is using "equidistributed curves";

The other is "supreme curves".

2). $d_{Th}(\cdot,\cdot)$ is asymmetric : $\exists\ \rho_0 \neq \rho_1$ $d_{Th}(\rho_0, \rho_1) \neq d_{Th}(\rho_1, \rho_0)$.

② Higher Teichmüller spaces

Replacing $PSL(2,\mathbb{R})$ by general Lie grp G, a higher Teichmüller space is a connected component of $\mathrm{Hom}(\pi_1(S), G)/\sim_G$ consisting of discrete and faithful reps.

example G = adjoint grp of split real Lie grp (eg $G = PSL(n,\mathbb{R})$ ...)

This is called the **Hitchin components**.

When $G = PSL(n,\mathbb{R})$, denote as $\mathcal{H}_n(S) \subseteq \mathrm{Hom}(\pi_1(S), PSL(n,\mathbb{R}))/\sim$

Rmk 1) $\mathcal{H}_n(S) \underset{\text{homeo}}{\simeq} \mathbb{R}^{(n^2-1)(2g-2)}$

$\mathcal{H}_2(S) = T(S)$

When $n>2$, $T(S) \subseteq \mathcal{H}_n(S)$ embedded copy of $T(S)$

$\rho \xrightarrow{i} i\circ\rho$

$i: SL(2,\mathbb{R}) \to SL(n,\mathbb{R})$ irreducible reps

This copy of $T(S)$ is called the **Fuchsian locus** in $\mathcal{H}_n(S)$.

2) $\mathcal{H}_n(S)$ are expected to parametrize geometric structures.

$n=3$: $\mathcal{H}_3(S)$ parametrizes convex real projective structures.

Rmk $\exists$ other types of higher Teich spaces :

When G = Hermitian Lie grp, these are maximal representations.

③ Metrics on higher Teich spaces

First Introduce length functions and entropy :

For $\rho \in Hn(S)$, consider lift of $\rho(\gamma)$ in $SL(n,\mathbb{R})$.

List $\lambda(\rho(\gamma)) = (\lambda_1(\rho\gamma) \cdots \lambda_n(\rho\gamma))$ in "$>$" order.

moduli of eigenvalues

$$L_\rho^\varphi(\gamma) := \log \frac{\lambda_1(\rho\gamma)}{\lambda_2(\rho\gamma)} \quad \text{1-st simple root}$$

$$L_\rho^\varphi(\gamma) := \log \lambda_1(\rho\gamma) \quad \text{spectral radius}$$

$\Big\}$ φ - length

Let

$$h^\varphi(\rho) := \lim_{T\to\infty} \frac{\log \#\{\gamma \in \pi_1(S) \mid l_\rho^\varphi(\gamma) \le T\}}{T}$$

exponential growth

be the φ - length topological entropy of $\rho \in Hn(S)$

To generalize Thurston / Riemannian metric $\langle \cdot , \cdot \rangle_p$ on $T(S)$, Thm (Bridgeman - Canary - Labourie - Sambarino, 2015)

Let $\{\rho_t\} \subseteq H_n(S)$, let $R_T^{\varphi}(\rho) := \{\gamma \in \pi(S) \mid l^{\varphi}_{\rho_0}(\gamma) \leq T\}$

Define the **φ- Intersection number**:

$$I^{\varphi}(\rho_0, \rho_t) = \lim_{T\to\infty} \frac{1}{\# R_T^{\varphi}(\rho_0)} \sum_{\gamma \in R_T^{\varphi}(\rho_0)} \frac{l^{\varphi}_{\rho_t}(\gamma)}{l^{\varphi}_{\rho_0}(\gamma)}$$

Similar to "equidistributed curves" and link to Bowen Margulis measure.

Then $\langle \cdot, \cdot \rangle_{\rho}^{\varphi}$: $TH_n(S) \times TH_n(S) \to \mathbb{R}$

$$\underbrace{\| \partial_t \rho_t |_{t=0} \|^2}_{\text{Variation along } \{\rho_t\}} (\rho_0) = \partial_t^2 \left(I^{\varphi}(\rho_0, \rho_t) \cdot \frac{h^{\varphi}(\rho_t)}{h^{\varphi}(\rho_0)} \right) \Big|_{t=0}$$

is a Riemannian metric, called the **(φ-) Pressure metric**.

To generalize Thurston's Finsler metric on $T(S)$,

Thm (Carvajales - D - Pozzetti - Wienhard, 2022)

$$d^{\varphi}_{Th} : \mathcal{H}_n(S) \times \mathcal{H}_n(S) \longrightarrow \mathbb{R}_{\geq 0}$$

$$d^{\varphi}_{Th}(\rho_0, \rho_1) := \log \sup_{\gamma \in \pi_1(S)} \frac{l^{\varphi}_{\rho_1}(\gamma)}{l^{\varphi}_{\rho_0}(\gamma)} \cdot \frac{h^{\varphi}(\rho_1)}{h^{\varphi}(\rho_0)}$$

is a Finsler metric on $\mathcal{H}_n(S)$.

metric Anosov

Idea of pfs:

(1) $\rho_0 \in \mathcal{H}_n(S) \longrightarrow$ construct flow $\underline{\phi_t^{\rho_0, \varphi}} \curvearrowright U\Gamma^{\rho_0, \varphi}$ "unit tangent bundle" for ρ.

Periods of $\phi_t^{\rho_0, \varphi}$ are $l^{\varphi}_{\rho_0}(\gamma)$ associated to $\gamma \in \pi_1(S)$.

(2) Reparametrize $\phi^{\rho_0} := \phi_t^{\rho_0, \varphi}$ to encode information for ρ_1:

$\exists$ $\underline{f_{\rho_1}} : U\Gamma^{\rho_0, \varphi} \longrightarrow \mathbb{R}_{\geq 0}$, reparametrization function for ρ_1

s.t. $\int_\gamma f_{\rho_1} = l^{\varphi}_{\rho_1}(\gamma)$

$\rho_0 \xrightarrow{f_{\rho_1}} \rho_1$

③ Thermodynamic formalism:

Define $$I^{\varphi}(\rho_0, \rho_1, m) := \int_{\mathcal{UP}^{\rho_0,\varphi}} f_{\rho_1}\, dm$$

m - intersection number

where m is a prob measure on $\mathcal{UP}^{\rho_0,\varphi}$.

Define $$J^{\varphi}(\rho_0, \rho_1, m) := \frac{h^{\varphi}(\rho_1)}{h^{\varphi}(\rho_0)} I^{\varphi}(\rho_0, \rho_1, m)$$

m - renormized intersection number

key (BCLS)

$$J^{\varphi}(\rho_0, \rho_1, m^{BM}_{\phi^{\rho_0}}) \geq 1 \; ;$$

Also "=" holds $\Leftrightarrow$ $l^{\varphi}_{\rho_0}(\gamma) = l^{\varphi}_{\rho_1}(\gamma)$ $\overset{\text{Rigidity result}}{\Longleftrightarrow}$ $\rho_0 = \rho_1$ (in $\mathcal{H}_n(S)$) $(*)$.

For us,
$$\sup_m J^{\varphi}(\rho_0, \rho_1, m) \geq J^{\varphi}(\rho_0, \rho_1, m^{BM}_{\phi^{\rho_0}}) \geq 1.$$
$$\| $$
$$\sup_{\gamma} J^{\varphi}\left(\rho_0, \rho_1, \frac{\delta_{\gamma}}{l_{\rho_0}(\gamma)}\right) = \sup_{\gamma} \int_{\mathcal{UP}^{\rho_0,\varphi}} f_{\rho_1} \frac{d\delta_{\gamma}}{l_{\rho_0}(\gamma)}$$
$$= \sup_{\gamma} \frac{l^{\varphi}_{\rho_1}(\gamma)}{l^{\varphi}_{\rho_0}(\gamma)}$$

Also $d_{Th}(\rho_0, \rho_1) = 0 \Rightarrow (*)$ $\square$