Boundary of Groups

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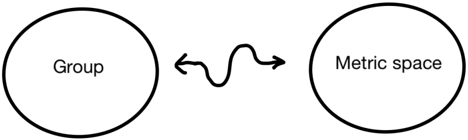
February 2023

Geometric Group Theory: using geometric, topological and dynamical methods to study algebraic properties of groups.

$$G = \langle s_1, s_2, \ldots \mid r_1, r_2 \ldots \rangle$$



Useful spaces:

1. Cayley graph:
   - vertices: $g \in G$.
   - edges: (g, gs), where s is a generator.
2. Surfaces and manifolds, other complexes.
3. **Word Problem**: Find a uniform test or mechanical procedure (i.e. an algorithm) which, given a word $w = s_{i_1} s_{i_2} s_{i_3} \ldots$, can decide whether $w = 1$.

Canonical equivalence classes: quasi-isometry equivalence

Different Cayley graphs of finitely generated groups are quasi-isometric to each other.

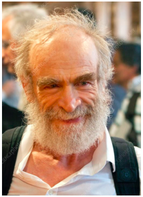
Definition

Let (X_1, d_1) and (X_2, d_2) be metric spaces. A map $\Phi\colon X_1 \to X_2$ is called a (q, Q)-*quasi-isometric embedding* if there exist constants $q \geq 1$ and $Q \geq 0$ such that for all $x, y \in X_1$

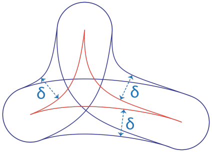
$$\frac{1}{q} d_1(x, y) - Q \leq d_2(\Phi(x), \Phi(y)) \leq q d_1(x, y) + Q.$$

"If a group G is quasi-isometric to a Euclidean plane, then G is virtually $\mathbb{Z}^2$*."*

Curvature in general metric spaces: Gromov hyperbolicity.



δ-hyperbolic space: there exists a constant δ such that every triangle is δ-thin. A group is δ-hyperbolic if its Cayley graph is δ-hyperbolic.



- Free groups of finite rank.
- Fundamental groups of closed surfaces of genus at least 2; Fundamental groups of closed, hyperbolic manifolds of higher dimensions.

This idea by Gromov led to a major breakthrough in geometric and combinatorial group theory in the following sense:

Gromov 87:

- solvable word problem.
- Finitely presented. If torsion free, then finite cohomological dimension.
- A generic group is hyperbolic.

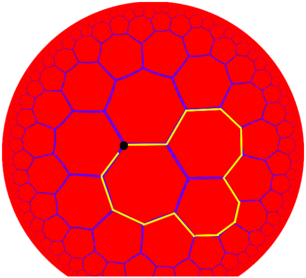


Figure: A trivial word is a loop in the Cayley complex.

Boundary in hyperbolic groups: the space of all directions

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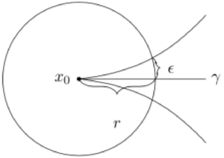
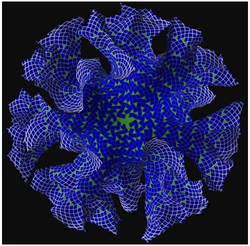
Definition

Let X be the Cayley graph of a hyperbolic group G.
Elements: infinite geodesic rays emanating from the base-point.
Equivalence class: fellow traveling.
Cone topology: $\mathcal{N}(\gamma, t, \epsilon) = \{\sigma \in \partial X | d(\sigma(t), \gamma(t)) < \epsilon\}$.

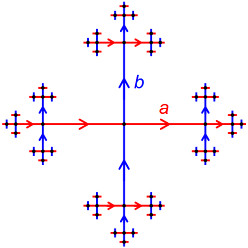


We call this topological space the Gromov boundary of G. The boundary is well-defined with respect to the associated group, compact, and metrizable (Gromov, 87).

Gromov boundary



$\partial\mathbb{H}^2 = S^1$



$\partial F_2 = \Sigma C$

Properties of the Gromov boundary:

- Group-invariant and metrizable;
- Large: different geodesic rays and different sample paths in a random walk end up at different points in the Gromov boundary.

Canon's Conjecture: If G is hyperbolic and ∂G is a 2-sphere, then G is virtually Kleinian.

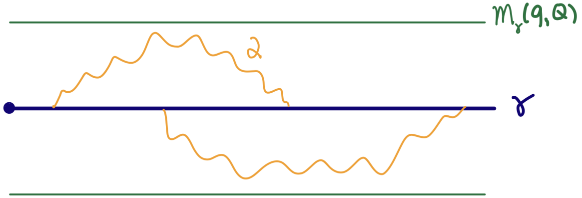
What about non-hyperbolic groups?

image credit: Andrew Yeghnazar

- Consider all directions: visual boundary of CAT(0) spaces, not a group invariant (Croke-Kleiner 00, Qing 16).
- Consider only the directions behaving like directions in Gromov hyperbolic space: Morse boundary, a *small* set of directions in some sense.(Charney-Sultan 13, Cordes 15)

Sublinearly-Morse directions

Space: $(X, \mathfrak{o})$ is a proper, geodesic space, with a fixed base-point $\mathfrak{o}$.
A geodesic ray γ is Morse if there exists a family of *gauge* functions $m_\gamma(q, Q) \colon \mathbb{R}^2 \to \mathbb{R}$ such that any (q, Q)-quasi-geodesic segment α with endpoints on γ is in a $m(q, Q)$-neighbourhood of γ.



Consequences:

- Quasi-isometry invariant.
- Fellow travel implies uniform fellow travel.

Fix a sublinear function $\kappa(t)$. Let $\|x\| = d(\mathfrak{o}, x)$.

A κ–neighbourhood around a quasi-geodesic γ is a set of point x

$$\mathcal{N}_\kappa(\gamma, n) := \{x \mid d(x, \gamma) \leq n \cdot \kappa(\|x\|)\}$$

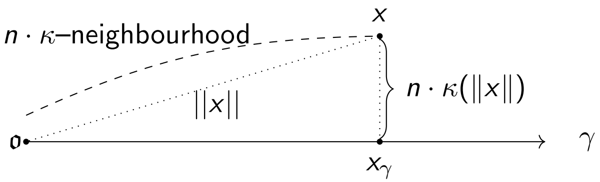
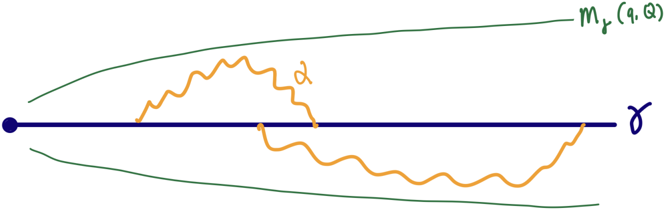


Figure: A κ–neighbourhood of γ

Definition of κ-Morse directions

A quasi-geodesic ray γ is κ–Morse if there exists a family of functions $m_\gamma(q, Q) : \mathbb{R}^2 \to \mathbb{R}$ such that any (q, Q)-quasi-geodesic segment α with endpoints on γ is in $\mathcal{N}_\kappa\big(\gamma, m_\gamma(q, Q)\big)$.



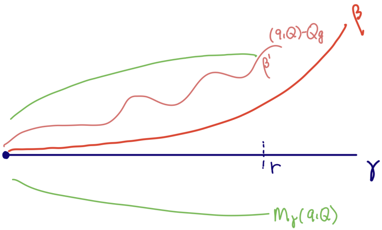
Consider all κ-Morse quasi-geodesic rays in $(X, \mathfrak{o})$. we say $\alpha \sim \beta$ if α, β sublinearly fellow travel, i.e.

$$\lim_{t \to \infty} \frac{d(\alpha(t), \beta(t))}{t} = 0.$$

The equivalence classes are elements of $\partial_\kappa X$.

Topology

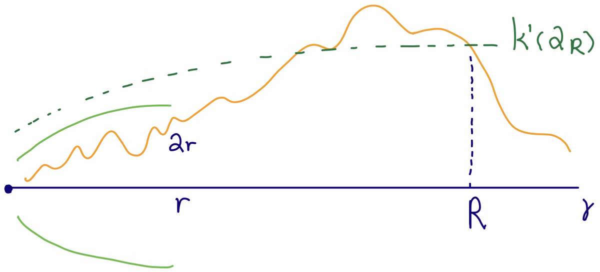
- The cone topology on all κ-Morse geodesic rays does not produce a QI-invariant space (Cashen, 19).
- Check the associated fellow traveling property for all quasi-geodesic rays for a bounded set of quasi-constants.



Equivalent Definition: *"sublinear fellow travel implies uniform sublinear fellow travel".*

A quasi-geodesic ray γ is κ–Morse if there exists a family of functions $m'_\gamma(q, Q) : \mathbb{R}^2 \to \mathbb{R}$ such for any sublinear function κ' and for any $r > 0$, there exists R such that for any (q, Q)-quasi-geodesic segment α we have that:

$$d(\alpha_R, \gamma) < \kappa'(R) \Longrightarrow \alpha|_r \in \mathcal{N}_\kappa\big(\gamma, m'_\gamma(q, Q)\big).$$



Let $\partial_\kappa X$ denote the equivalence class of all κ-Morse quasi-geodesic rays equipped with the coarse cone topology.

Theorem (Q-Rafi 19, Q-Rafi-Tiozzo 21)

Let X be a proper, geodesic metric space. Fix a sublinearly function κ, the $\partial_\kappa X$ is a topological space that is invariant under quasi-isometry of X, and metrizable.

In particular, for every finitely generated group G and for every sublinear function κ, $\partial_\kappa G$ is well defined up to homeomorphism and it is a metrizable space.

Examples:

- $\mathbb{Z}^2$: $\partial_\kappa \mathbb{Z}^2 = \emptyset$.
- $\mathbb{H}^2$: every (quasi-)geodesic ray is Morse and hence sublinearly Morse: $\partial_\kappa \mathbb{H}^2 = S^1$.
- $\mathbb{Z} \star \mathbb{Z}^2$

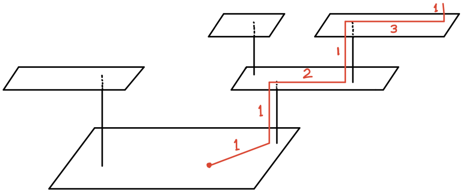
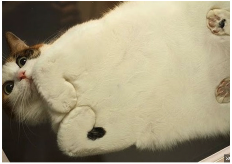
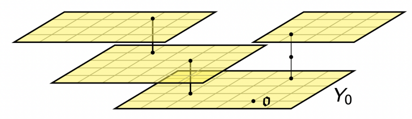
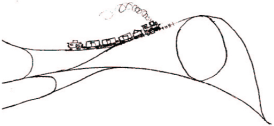


Figure: A tree of flats.

thickness of the neighborhood in the n^{th} flat: $\sim n$
distance of the n^{th} flat to the origin: $\sim n^2$.

Specific classes of groups and spaces of negative curvature feature:

CAT(0) groups 	Relatively hyperbolic groups 
Mapping class group  Bill Thurston	

CAT(0) Space

Non-positively curved spaces (Cartan, Aleksandrov and Toponogov).

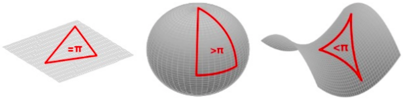


Figure: Model spaces, real 2-dimensional Riemannian manifold.

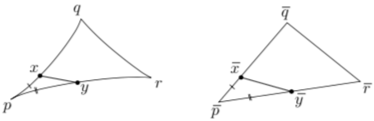


Figure: non-fat triangles

Examples include right-angled Artin groups, some braid groups and right-angled Coxeter groups, etc. CAT(0) groups have solvable word problems.

Evidence of genericity of sublinear directions: Patterson-Sullivan measures

Definition

More generally, let $G \curvearrowright X$ be any properly discontinuous action of a countable group on a metric space. For $\mathfrak{o} \in X$ let $B_R(\mathfrak{o})$ be the ball of radius R in X centered at $\mathfrak{o}$.
The (possibly zero or infinite) quantity

$$\delta_X(G) = \limsup_{R \to \infty} R^{-1} \log |B_R(\mathfrak{o}) \cap G \cdot \mathfrak{o}|$$

is called the critical exponent of the action $G \curvearrowright X$.

For $x, y \in X$ and $\zeta, \alpha \in \partial_{\text{vis}} X$ define the Busemann function

$$\beta_\zeta(x, y) = \lim_{z \to \zeta} d(x, z) - d(y, z).$$

A $\delta(G)$-*conformal density* for $G \curvearrowright X$ is an absolutely continuous family of finite Borel measures $\nu_x, x \in X$ on the limit set $L(G)$ such that

$$d\nu_x / d\nu_y(\zeta) = \exp(\delta(G)\beta_\zeta(y, x))$$

and $g\nu_x = \nu_{g^{-1}x}$ for any $x, y \in X$ and $g \in G$.

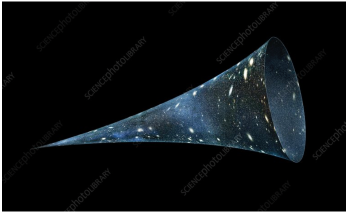


Figure: A fraction of the directions.

When $G \curvearrowright X$ is properly discontinuous and cocompact there is a unique conformal density for $G \curvearrowright X$; the measure ν_0 is called the Patterson-Sullivan measure.

Wenyuan Yang 22: Morse boundary is measure zero in the visual boundary of CAT(0) spaces with Patterson-Sullivan measure.

Theorem (Gekhtman-Q.-Rafi 22)

Let $G \curvearrowright X$ be a countable group acting properly discontinuously by isometries on a geodesically complete rank-1 CAT(0) space X. Assume the action is cocompact. Let ν be the Patterson-Sullivan measure on the visual boundary of X. Then ν gives measure zero to the complement of sublinearly Morse directions.

Application: random walk on groups and Poisson boundaries

Let $\langle S \rangle$ be a symmetric generating set with a probability distribution μ. A *random walk* is a process on a group G (or its Cayley graph) where sample paths are $s_{r_1} s_{r_2} s_{r_3} \ldots,\ s_{r_i} \in \langle S \rangle$.

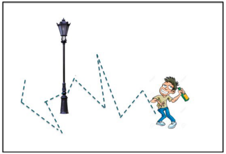
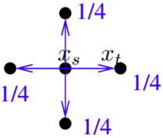


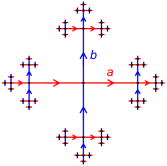
Figure: A random walk.

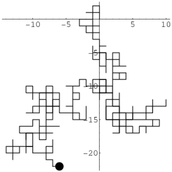
Consider the set of all sample paths. We say two sample paths are equivalent if they coincide from some step onward.

- In hyperbolic groups, random walks tend to the boundary in linearly speed. In $\mathbb{Z}^2$ random walks are recurrent.



Random walk on the free group vs. random walk on abelian group.





Poisson boundary

The asymptotic behaviour of all random walks is encoded by the *Poisson boundary*.

Definition

Given a finitely generated group and a probability measure μ with finite support, its *Poisson boundary* is the maximal measurable set to which almost all sample paths converge, with hitting measure ν arising from μ.

A natural problem is to determine when this space is trivial and, if it is not, to exhibit a geometric model.

Kaimanovich 96: Let G be a hyperbolic group, then Gromov boundary is a model for its associated Poisson boundary.

Relative hyperbolic groups

Definition

Relative hyperbolic group: A group G is *relatively hyperbolic* with respect to a subgroup H, if

I. after contracting the Cayley graph of G along H-cosets, the resulting graph equipped with the usual graph metric becomes a δ-hyperbolic space, and
II. Pairs of quasi-geodesics sharing the endpoints have bounded coset penetration property.

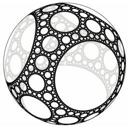
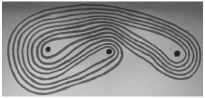


Image created by C. McMullen, A. Mohammadi and H. Oh

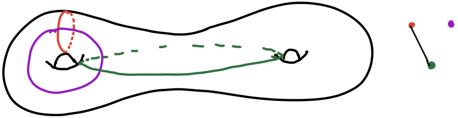
Mapping class group: Let S be a closed, oriented surface and consider

$$Map(S) := Homeo^+(S)/Isotopy$$



Thurston's drawing 1971.

Mapping class groups are in general not hyperbolic or relatively hyperbolic, not CAT(0). It acts nicely on curve graphs:



Curve graphs are infinite diameter and hyperbolic.

Theorem (Q-Rafi-Tiozzo, 21)

Let G be either

- *a non-elementary relatively hyperbolic group, or*
- *a mapping class group,*

and let μ be a probability measure whose support is finite and generates G as a semigroup. Then for $\kappa(t) = \log(t)$, we have:

1. *Almost every sample path (w_n) converges to a point in $\partial_\kappa G$;*
2. *The κ-Morse boundary $(\partial_\kappa G, \nu)$ is a model for the Poisson boundary of (G, μ) where ν is the hitting measure associated to the random walk driven by μ.*

Proof idea: logrithmic excursions in projection system (Sisto-Taylor, 19).

The theorems address open questions regarding the invariance of Poisson boundaries posed by Kaimanovich.

- Relative hyperbolic groups
- Curve complex of subsurfaces in mapping class group.
- Hierarchically hyperbolic groups.

Let G be a group and let $(\mathcal{S}, Z_0, \{\pi_Z\}_{Z\in\mathcal{S}}, \pitchfork)$ be a projection system on G. Let (w_n) be a random walk on G. Then there exists $C \geq 1$ so that, as n goes to ∞,

$$\mathbb{P}\Big(\sup_{Z\in\mathcal{S}} d_Z(1, w_n) \in [C^{-1}\log n, C\log n]\Big) \to 1$$

Inhyeok Choi 22: any finitely generated group with two independent contracting isometries.

Bonus theorem: Poisson boundaries of CAT(0) groups

Theorem (Karlsson-Margulis 99, Nevo-Sageev 11)

Let G acts geometrically on a CAT(0) space X. The visual boundary of X is a topological model for the Poisson boundary of G.

Theorem (Gekhtman-Qing-Rafi, 22)

For any rank-1 CAT(0) group G, there exists κ such that the κ–contracting boundary of the group a topological model for its Poisson boundary.

Idea: frequently contracting rays

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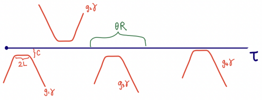
A unit speed parametrized geodesic ray $\tau \colon [0, \infty) \to X$ is (N, C)–frequently contracting for constant $N, C > 0$ if:
For each $L > 0$ and $\theta \in (0, 1)$ there is an $R_0 > 0$ such that for $R > R_0$ and $t > 0$ there is an interval of time $[s - L, s + L] \subset [t, t + \theta R]$ and an N –contracting geodesic γ such that,

$$u \in [s - L, s + L] \Longrightarrow d(\tau(u), \gamma) \leq C.$$

That is, every subsegment (of τ) of length θR contains a segment of length $2L$ that is C-close to an N−contracting geodesic γ.



Important detail: the minimal κ-function varies with different groups!

Theorem (Q.-Tiozzo 19, Q.-Rafi-Tiozzo 21)

the Poisson boundary can be identified with $\partial_\kappa G$ for the following groups.

- *Right-angled Artin groups,* $\kappa(t) = \sqrt{t \log t}$.
- *Relative hyperbolic groups,* $\kappa(t) = \log t$
- *Mapping class groups,* $\kappa(t) = \log t$.

Other properties of the sublinearly Morse boundaries of CAT(0) spaces:

- Visibility space: there exists a geodesic line that connects two classes in $\partial_\kappa X$ (Zalloum).
- Minimality: for each $\mathbf{a} \in \partial_\kappa X$, $G \cdot \mathbf{a}$ is dense in $\partial_\kappa X$ (Q.-Zalloum).
- Sublinearly Bilipschitz equivalence invariant (Pallier -Q.).
- superlinear divergence (Murray-Q.-Zalloum).

Sublinear BiLipschitz Equivalence (Cornulier, 09)

Definition (θ-SBE)

Let $(X, \mathfrak{o})$ and $(Y, \mathfrak{o})$ be proper geodesic metric spaces with basepoints. Let $L \geqslant 1$ be a constant, and let θ be a sublinear function as before. We say that $\Phi : X \to Y$ is a (L, θ)-sublinear biLipschitz equivalence (θ-SBE) if

$$\begin{aligned}\frac{1}{L} d(x_1, x_2) - \theta(\max(\|x_1\|, \|x_2\|)) &\leqslant d(\Phi(x_1), \Phi(x_2))) \\ &\leqslant L d(x_1, x_2) + \theta(\max(\|x_1\|, \|x_2\|))\end{aligned}$$

and $Y = \mathcal{N}_\theta(\Phi(X), D)$ for some $D \geqslant 0$.

Theorem

Gromov hyperbolicity admits a characterization in terms of asymptotic cones, it is an SBE-invariant property.

Goal: Find other properties that are SBE invariant.

Theorem (Pallier-Q. 23')

Let Φ *be an* θ*-SBE between two proper geodesic spaces* X *and* Y*. Suppose* $\theta \preceq \kappa$*. Then* Φ *induces a homeomorphism* $\Phi_\star \colon \partial_\kappa X \to \partial_\kappa Y$*.*

Key idea: the image of a κ-Morse quasi-geodesic ray under a θ-SBE is still a κ-Morse set.

Application: Asymptotic cones of right-angled Coxeter groups

Definition

Let G be a simplicial graph on a set of S elements, then

$$W = \langle s \in S : s^2 = 1, \text{ and } st = ts \text{ if } (s,t) \in E(G) \rangle$$

is a right-angled Coxeter group.

Examples

- If G is the complete graph on S, then $W = (\mathbb{Z}/2\mathbb{Z})^{|S|}$.
- If G has no edges, then $W = \mathbb{Z}/2\mathbb{Z} \star ... \star \mathbb{Z}/2\mathbb{Z}$, the free product of $|S|$ copies of $\mathbb{Z}_2$.

Example (Behrstock, 2015)

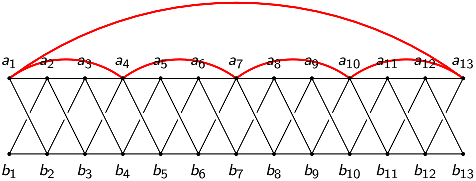


Figure: The graph for the group W vs W_{13}

Casals-Ruiz, Hagen and Kazachkov (working draft): W and W_{13} have unique asymptotic cones.

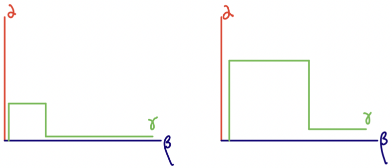
Question: Are W_{13} and W sublinearly biLipschitz equivalent?
Answer: No. Since $\partial_\kappa W_{13}$ and $\partial_\kappa W$ are not homeomorphic, by the main theorem, the associated spaces cannot be SBE.

Compactification of the sublinearly Morse boundaries:

Let α, β be two quasi-geodesic rays in $(X, \mathfrak{o})$. We say that $\alpha \preceq \beta$ if any only if there exists a pair of constants (q, Q) such that for every $r > 0$ there exists a (q, Q) quasi-geodesic ray γ^r such that

$$\gamma^r|_r = \alpha|_r \text{ and } \gamma \text{ eventually coincides with } \beta.$$

We say $\alpha \sim \beta$ if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. Let $P(X)$ denote the equivalence classes of all quasi-geodesic rays under $\sim$.



Example: Baumslag Solitar groups.

Theorem (Q.-Rafi, 23)

Let G be a relative hyperbolic group with respect to subgroups that are flat-like. Then there exists a boundary ∂G that is

- *QI-invariant;*
- *Compact;*
- *Metrizable;*
- *Almost every sample path (w_n) converges to a point in ∂G;*
- *Contains $\partial_\kappa G$ as a topological subspace for each κ;*
- *A topology model for the Poisson boundaries;*
- *Homeomorphic to a natural Bowditch boundary.*

Thank you for your time!