A mapping class group-equivariant deformation retraction of Teichmüller space

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Talk Outline

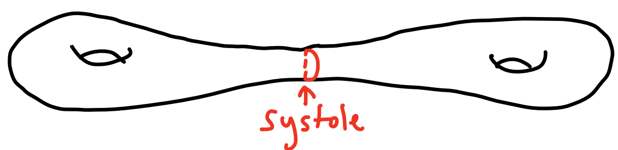
- Systoles, topological Morse functions
- Thurston's deformation retraction
- The Steinberg Module
- A further deformation retraction
- Schmutz's cells and duality
- Open questions

Definitions

Let S_g be a closed, orientable surface of genus g without marked points, and $\mathcal{T}_g$ be its Teichmüller space. Once a point in $\mathcal{T}_g$ has been chosen, S_g will refer to the corresponding hyperbolic surface.

Γ_g - mapping class group of S_g

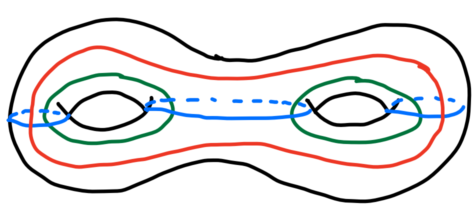
A systole on S_g is a geodesic with length less than or equal to that of any other geodesic on S_g



Important observation - a pair of systoles can intersect in at most 1 point.

A set of geodesics "fills" S_g if the complement is a set of polygons

e.g.



Systole function

$f_{sys} : \mathcal{T}_g \to \mathbb{R}_+$, $x \mapsto$ length of systole at x

$\mathrm{Sys}(C) :=$ The subset of T_g on which C is the set of systoles

Let c be a curve on S_g.

$L(c) : T_g \to \mathbb{R}_+$, $x \mapsto$ length of c at x

A length function is a positive linear combination of such functions

Length functions are analytic functions satisfying many convexity properties

Lemma (I.I) When the curves in C fill, $\mathrm{Sys}(C)$ is a connected, open subset of an embedded submanifold of T_g, with compact closure.

Thurston Spine P_g - The set of all points of T_g at which the systoles fill. This is a CW complex, with cells of the form $\mathrm{Sys}(C)$.

Akrout - f_{sys} is a topological Morse function

Claim - the Thurston spine is the Morse-Smale complex of f_{sys}

Thurston's deformation Retraction

Key Lemma - Let C be any collection of curves on a surface that do not fill. Then at any point of T_g there are tangent vectors that simultaneously increase the lengths of all the geodesics representing curves in C.

Proof - Using Lipschitz maps.

Independent proofs due to Bers, Riera, I.I. , Parlier

2 steps

- Construct a Γ_g-equivariant isotopy ϕ_t of $\mathcal{T}_g$ into a regular neighbourhood of the Thurston spine P_g.
  This is done using a flow whose existence is guaranteed by the key lemma
- Use the structure of the regular neighbourhood to retract the rest of the way onto P_g.

1st step, the flow

Define $P_{g,\epsilon}$ to be the subset of $\mathcal{T}_g$ for which the set of geodesics whose length is within ε of the length of a systole fill.

Each $P_{g,\varepsilon}$ has compact closure modulo the action of Γ_g.

If $\mathcal{N}$ is a regular open neighbourhood of P_g $\exists$ ε s.t $P_{g,\epsilon} \subset \mathcal{N}$

At a point x of $\mathcal{T}_g \setminus P_g$, let $C(x)$ be a set of shortest geodesics at x.

When the curves in $C(x)$ don't fill, use key lemma to construct v.f. X_C with the property that the length of every curve in C is increasing in the direction of X_C

Choices can be made in such a way that X_C is Γ_g-equivariant and smooth away from where C changes

Smoothing off

Define $U_{C(\epsilon)} := \{ x \in \mathcal{T}_g \mid C \text{ is the set of geodesics at } x \text{ with length at most } f_{sys}(x) + |C| \varepsilon \}$

ε small enough $\Rightarrow \underbrace{\{U_{C_i}}_{\text{covers } \mathcal{T}_g} \mid C_i \text{ is a finite set of curves on } S_g \}$

Γ_g-equivariant partition of unity $\{\lambda_{C_i}\}$ subordinate to $\{U_{C_i}\}$

Averaging over the $\{X_{C_i}\}$ does not create zeros.

We will see that the value of the VF on an arbitrarily small nbhd of P_g will not matter. Don't worry about this for now.

Arbitrarily define $X_{C_i} = 0$ if C_i fills.

Choose ε sufficiently small s.t. a VF X_{C_i} can only be zero within a narrow regular neighbourhood of P_g

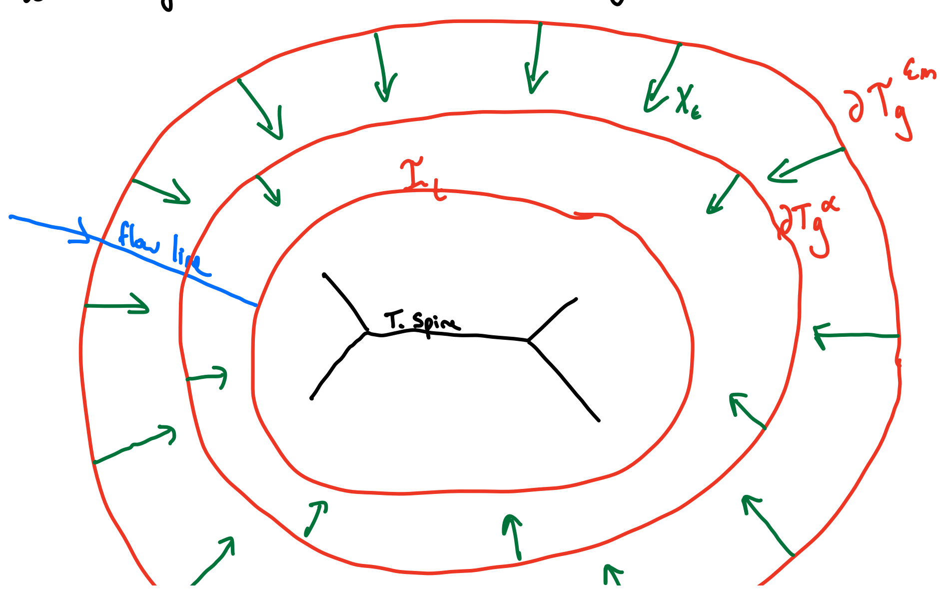
Possible because P_g/Γ_g compact

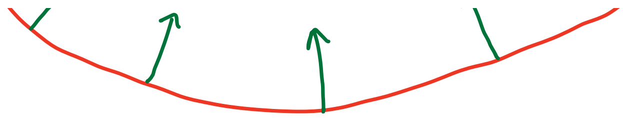
Call the resulting VF X_ε

Next, take a set K, $P_g \subset K \subset \mathcal{T}_g$, K/Γ_g compact

For simplicity, we will take the thick part, $K = \mathcal{T}_g^{\varepsilon_m}$,
$\varepsilon_m :=$ Margulis constant

Define isotopy $\phi_t : \mathcal{T}_g^{\varepsilon_m} \longrightarrow \mathcal{I}_t$,
$x \mapsto$ image at time t under flow generated by X_ε





By construction when X_ε is nonzero, it is pointing inwards on the boundary of a level set T_g^α of f_{sys}

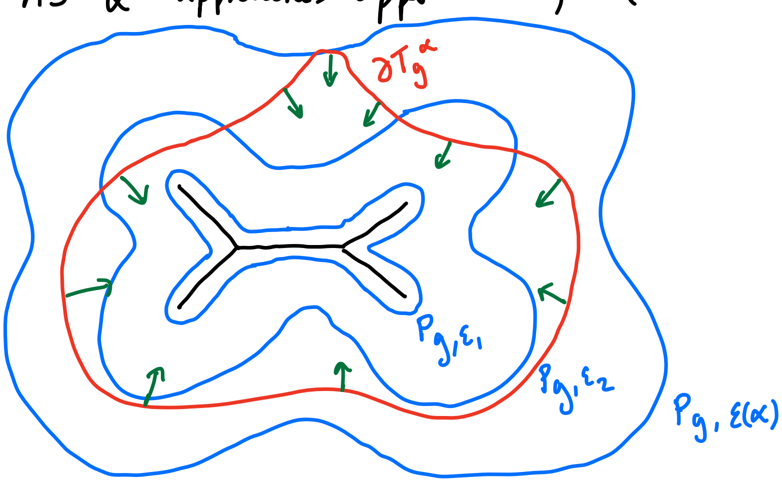
Each α-thick part, T_g^α is invariant under the flow!

Define $\beta : T_g \to \mathbb{R}_+$, $x \mapsto$ smallest real number r s.t the set of geodesics with length no more than $f_{sys}(x) + r$ fill

$$\varepsilon(\alpha) := \max_{x \in T_g^\alpha} \beta(x)$$

$\varepsilon(\alpha)$ is nonincreasing with α

As α approaches upper bound, $\varepsilon(\alpha) \to 0$



Once a point flows into T_g^α, it cannot flow out of $P_{g,\varepsilon(\alpha)}$!

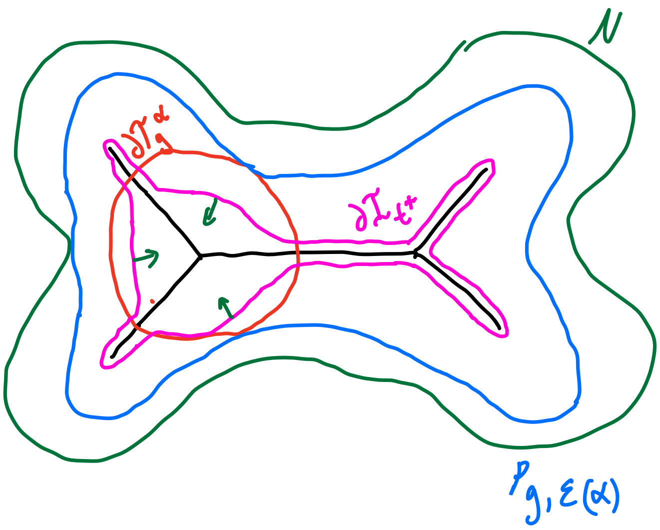
By compactness of $T_g^{\varepsilon_m}/\Gamma_g$, outside a small neighbourhood of P_g, rate of increase of f_{sys} along each flowline is bounded from below

$\Rightarrow$ after a finite time, each point in $T_g^{\varepsilon_m}$ is flowed into small neighbourhood of P_g on which $\|X_\varepsilon\|$ is small

If ε in the defn of X_ε is decreased with time, can flow into, and stay in, $P_{g,\varepsilon'}$ for arbitrarily small ε'.

Choose $t^* \in [0,\infty)$ such that $\phi_{t^*}(\Upsilon_g^{\varepsilon_m}) \subset N$

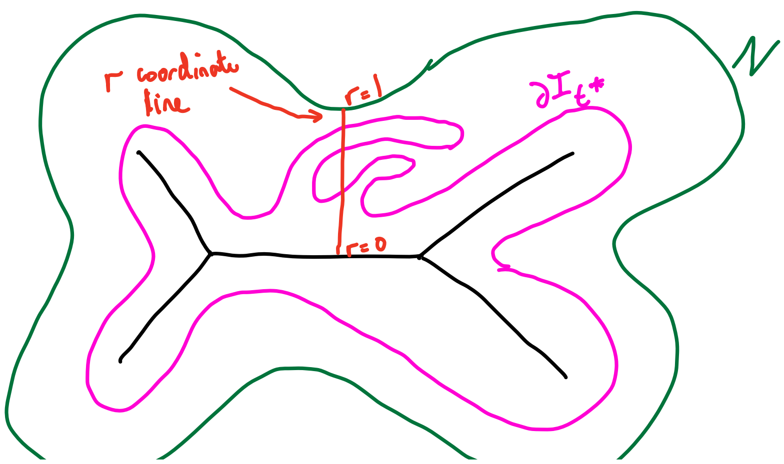
where N is a normal neighbourhood



2nd step, within the normal neighbourhood N

∂I_{t^*} must be connected, because $\partial \Upsilon_g^{\varepsilon_m}$ is.

∂I_{t^*} separates ∂N from P_g, and is smooth

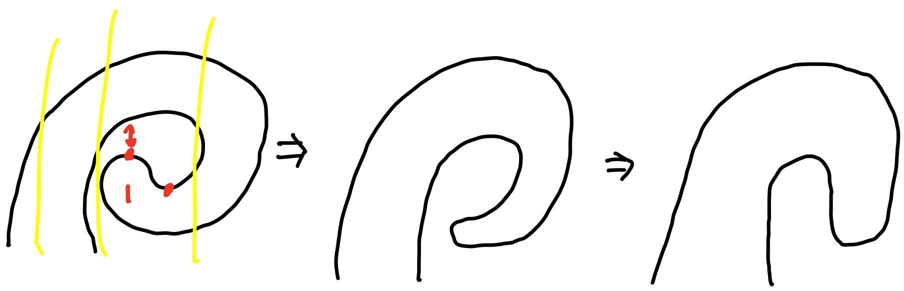


Need to either deform the normal coordinates on $\mathcal{N}$, or retract I_{t^*} to resolve this

Different approaches

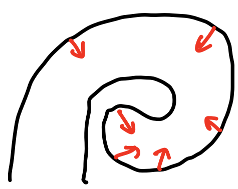
- $r|_{\partial I_{t^*}}$ by taking arbitrarily small deformation, can assume w.l.o.g only has points with nondegenerate Hessian (Theorem from Morse theory)

  Deform normal coordinate until enough critical points are cancelled out



- Alternatively, deform one r-coordinate line to intersect ∂I_t once, and extend this to a neighbourhood of r-lines.

- Use curvature



(messy to make rigorous)

The Steinberg Module

C_g - Harvey's complex of curves

Harer - C_g is homotopy equivalent to a wedge of spheres
$$\bigvee_i^{\infty} S^{2g-2}$$

Ivanov - $\partial T_g^{\varepsilon m}$ is contractible into $T_g^{\varepsilon m}$, and its boundary is homotopy equivalent to C_g

Since C_g admits an action of Γ_g, this gives the Steinberg module $St(\Sigma) := \tilde{H}_{2g-2}(C(\Sigma); \mathbb{Z})$ the structure of a Γ_g-module

A furthur deformation retraction

$$cd(G) := \sup\{n \in \mathbb{N} \mid H^n(G, M) \neq 0 \text{ for some module } M\}$$

Γ_g has finite index torsion free subgroups

Serre - Any finite index torsion free subgroup has the same cohomological dimension

$$vcd(\Gamma_g) := cd\left(\frac{\Gamma_g}{\text{torsion}}\right)$$

$vcd(\Gamma_g)$ gives a lower bound on the dimension of the image of a Γ_g-equivariant deformation retraction of T_g

Harer - Explicit deformation retraction achieving this lower bound for punctured surfaces.

Harer - $vcd(\Gamma_g) = 4g-5$

Theorem (I.I) - The Thurston spine of a closed, orientable surface of genus g deformation retracts onto a subcomplex of dimension equal to $4g-5$

Proof: Let $Sys(C)$ be a top dimensional cell of P_g

Let q be an interior point of $Sys(C)$

- $D(q) :=$ pre-image of q under the deformation retraction of T_g onto P_g
- $D(q)$ is a ball with boundary at ∞
- $D(q)$ intersects P_g in the single point q
- $\dim(D(q)) =$ codim of $Sys(C)$ in T_g
- Since $T_g^{\epsilon m}$ is invariant under Thurston's flow, $D(q)$ intersects $\partial T_g^{\epsilon m}$ transversely in a connected set.
- $D(q) \cap \partial T_g^{\epsilon m} := S^{\epsilon m}$, a sphere of dimension codim $Sys(C) - 1$

$\dim(Sys(C))$ can't be less than $4g-5$.
Suppose $\dim(Sys(C)) > 4g-5$.
Then $\dim(S^{thick}) < 2g-2$
S^{thick} is $\therefore$ contractible in ∂T_g^{thick}

$\Rightarrow D(q) \cap T_g^{\epsilon m}$ can be homotoped relative to its boundary S^{thick} into $\partial T_g^{\epsilon m}$

This homotopy moves q off P_g
$\therefore P_g$ must have an unmatched face.
Collapse in the unmatched face, and repeat the argument, until a subcomplex with dimension $4g-5$ is obtained. $\square$

Schmutz's Cells and Duality

Schmutz defined cell decompositions parametrised by length functions.

For surfaces without punctures / marked points, existence was not known

$\min(C)$ - the set of points in $\mathcal{T}_g$ at which length functions written as positive linear combinations of lengths of curves in C have their minima

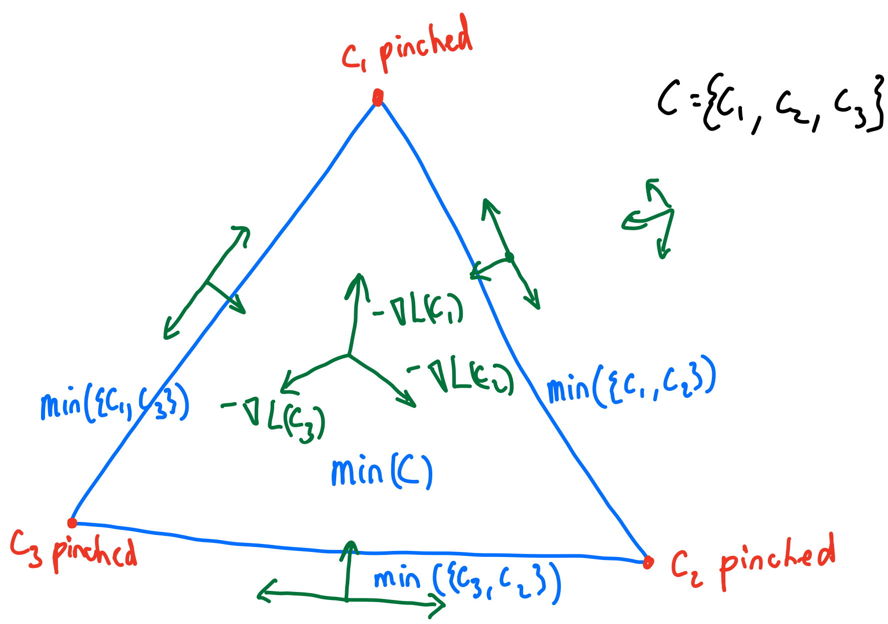
Lemma (Schmutz) - $\min(C)$ is nonempty iff the curves in C fill.

Lemma (Schmutz) - Let C be a set of geodesics that fills S_g. A point p of $\mathcal{T}_g$ is in $\overline{\min(C)}$ iff there does not exist a derivation in $T_p\mathcal{T}_g$ whose evaluation on each length function of a curve in C is strictly positive

Lemma (Schmutz) - Let $C = \{c_1, \ldots, c_k\}$, let $F(C) : \mathcal{T}_g \to \mathbb{R}_+^k$

$x \mapsto (L(c_1), \ldots, L(c_k))$

When the rank of the Jacobian of $F(C)$ is constant on $\min(C)$, the set $\min(C)$ is an open cell.



Theorem (I.I.) "Duality" between $\min(C)$ and $\mathrm{Sys}(C)$

Let $\mathrm{Sys}(C)$ be a cell of $\mathcal{P}_g$. Suppose there is a critical point p of f_{sys} contained in $\mathrm{Sys}(C)$. At p

$$\text{index of } f_{sys} \text{ at } p + \text{dimension of } \mathrm{Sys}(C) = \text{dimension of } \mathcal{T}_g$$

Moreover, p must be the unique point of intersection of $\mathrm{Sys}(C)$ with $\min(C)$, and

$$\text{index of } f_{sys} \text{ at } p = \text{dimension of } \overline{\min(C)} \text{ at } p.$$

Then $\{ \min(C) \mid \mathrm{Sys}(C) \text{ is a cell of } \mathcal{P}_g \}$ is a Γ_g-equivariant "set decomposition" of $\mathcal{T}_g$ dual to $\mathcal{P}_g$

Claim - this "set decomposition" is analogous to the cell decompositions for Teichmüller spaces of punctd surfaces used by Harer to show the existence of a Γ_g-equivariant deformation retraction for punctured surfaces.

Questions

There is a PL structure on $\min(C)$ with the help of which $\min(C)$ can be subdivided into cells

Question: How, where, why does the rank of $f(C)$ drop on $\min(C)$?

see also Corollary 3.2 of Schmutz "Riemann surfaces with shortest geodesic"

Another partial result - If C has no proper filling subsets, the rank of the Jacobian of $f(C)$ is constant on $\min(C)$

Every cell $Sys(C)$ of $\mathcal{P}_g$ can have at most 1 critical point of f_{sys}.

Question - does every cell contain a critical point?

On $Sys(C)$, the lengths of $\{\nabla L(c_i) \mid c_i \in C\}$ w.r.t the WP metric are about the same.

The tangent space to $Sys(C)$ contains all vectors onto which the projections of $\{\nabla L(c_i) \mid c_i \in C\}$ have the same lengths. Is this tangent space always contained in the convex hull of $\{\nabla L(c_i) \mid c_i \in C\}$?