Hitchin represenations and mimimal surfaces

Nathaniel Sagman (joint work with P.Smillie and V.Marković-P.Smillie)

University of Luxembourg

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Outline

- Hitchin representations and the Hitchin base

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- Minimal surfaces and Labourie's Conjecture

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- Proofs

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- Basic facts: $\mathbf{T}_g \ '_{C^1} \ \mathrm{R}^{6g \ 6}; \mathbf{T}_g$ has a complex structure, the mapping class group $MCG(\Sigma_g)$ acts properly discontinuously.

Higher Teichmüller spaces

- When unspecified, G is a semisimple real Lie group with no compact factors with finite center, i.e., $G = G_1$ G_m, with each G_i simple, non-compact type, and finite center.

Definition (Wienhard)

A higher Teichmüller space is a union of connected components of $\text{Rep}(G)$ consisting entirely of discrete and faithful representations.

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- $\mathrm{Hit}(G)$ is homeomorphic to $\mathbb{R}^{(2g-2)\dim(G)}$ (Hitchin, 1990), the Hitchin component is a higher Teichmüller space (Labourie, Fock-Goncharov, 2006), and MCG acts properly discontinuously (Labourie, 2006).

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Definition

The Hitchin base for $\mathrm{PSL}(n, \mathbb{C})$ is $B(S) = \bigoplus_{i=2}^{n} H^0(S, K^i) \simeq_{C^\infty} \mathbb{C}^{\dim G(g-1)}$.

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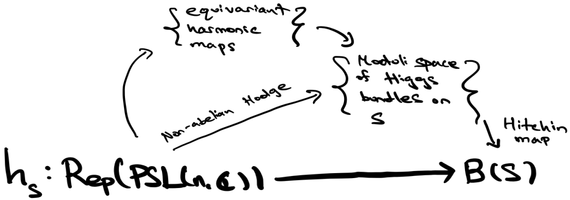
- A section of K^i is a tensor q locally of the form $q = q(z)dz^i$, $q(z)$ holomorphic.
- There is a surjective map



Hitchin section $b_S : B(S) \to \mathrm{Rep}(\mathrm{PSL}(n,\mathbb{C}))$ such that $b_S(B(S)) = \mathrm{Hit}(n)$.

Let

$$\mathfrak{t}^{\mathbb{C}} = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} : a_i \in \mathbb{C}, \sum a_i = 0 \right\}, \quad \text{(Cartan subalgebra)}.$$

A point in $B(S)$ is equivalent to

- a finite degree branched covering $S_B \to S$ (the cameral cover) whose deck group has a faithful representation $\rho : \mathrm{Deck}(S_B) \to S_n$ (the Weyl group).
- On S_B, a ρ-equivariant holomorphic 1-form with values in the trivial $\mathfrak{t}^{\mathbb{C}}$-bundle (satisfying a condition relative to the branch points).

The action of $\sigma \in S_n$ on the $\mathfrak{t}^{\mathbb{C}}$-bundle is

$$\sigma \cdot \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} a_{\sigma(1)} & & \\ & \ddots & \\ & & a_{\sigma(n)} \end{pmatrix}.$$

That is, we have a matrix of 1-forms on a cover S_B, whose entries are permuted by the Deck group action.

In one direction.

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- Let ${}_2, \dots, {}_n$ be the elementary symmetric polynomials, acting on by

$${}_2(\ ) = \sum_{j<k} {}_j\ {}_k, \quad {}_3(\ ) = \sum_{j<k<\ell} {}_j\ {}_k\ {}_\ell, \dots, \quad {}_n(\ ) = \prod_{j=1}^{n} {}_j.$$

More on the Hitchin base for $\mathrm{PSL}(n, \mathbb{C})$

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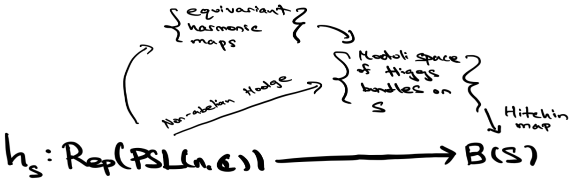
- ${}_i( )$ defines a holomorphic i-differential on S_B.
- For any $\in S_n$, ${}_i( ) = {}_i( )$. By equivariance, each ${}_i$ descends to a holomorphic i-differential q_i on S.

Hitchin's parametrization for $G = \mathrm{PSL}(n; \mathbb{R})$



- Once we've specified S_B; by (abelian) Hodge theory, is equivalent to an n-tuple of classes in $H^1(S_B; \mathbb{C})^n (= \mathrm{Rep}(\mathbb{C}^n))$ satisfying a linear relation.

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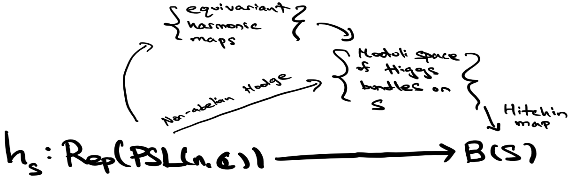

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- The Hitchin section b_S : B(S) ! Hit(n) associates the data of the cover S_B and the (abelian) section to a unique Hitchin representation.

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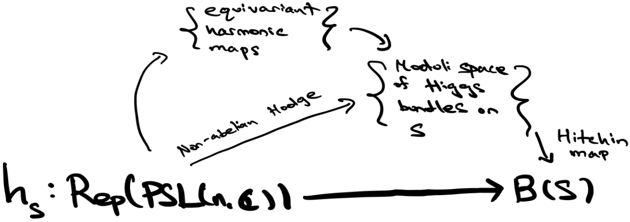

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Two issues with Hitchin's parametrization:

- No $\mathrm{MCG}(\Sigma_g)$ equivariance.
- In the induced complex structure, the inclusion $\mathbf{T}_g$! $\mathrm{Hit}(n)$ is not holomorphic.

- For general admissible G, let K be a maximal compact subgroup. Then $G{=}K$ admits a G-invariant metric , so that $(G{=}K; \ )$ is a symmetric space of non-compact type.

Minimal surfaces

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- Ex. for $G =$ PSL$(2;$ R$);$ $K =$ PSO$(2;$ R$)$ and $(G{=}K;\ \ )$ hyperbolic space.
- Given $[\ \ ]\ 2$ Rep(G), we consider C^2 maps $h : \tilde{\Sigma}_g\ \ !\ \ G{=}K$ that are -equivariant: for all $\ \ 2\ \ {}_1(\Sigma_g);\ h\ \ \ \ {}^{1} =\ \ (\ \ )h$.

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- Since is acting by isometries on $(G{=}K;\ )$, the pullback h dvol is $_1(\Sigma_g)$-invariant, and hence descends to a 2-form on Σ_g.
- The area of the map is

$$\mathrm{Area}(h) = A(h) = \mathrm{Z}_{\Sigma_g}\ h\ \mathrm{dvol}\ :$$

Definition

h is minimal if it is a critical point of $A;$ i.e., for all smooth variations h_t through C^2 -equivariant maps,

$$\frac{d}{dt} j_{t=0} A(h_t) = 0:$$

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Suppose lies in any known higher Teichmüller space. Then there exists an area minimizing -equivariant minimal map $h : \tilde{\Sigma}_g \ ! \ (G=K; \ ):$

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- Given any -equivariant minimal map h, there is a Riemann surface S such that $h : \tilde{S} \ ! \ (G{=}K; \ )$ is weakly conformal.
- **Fact:** for $G = \mathrm{PSL}(n; \mathrm{R})$, there exists a weakly conformal minimal map $h : \tilde{S} \ ! \ (G{=}K; \ )$ if and only if $h_S(\ ) = (0; q_3; : : : ; q_n) \ 2 \ {}^{n}_{i=2} H^0(S; K^i);$ i.e., the q_2 term vanishes (or $_2(\ ) = 0$).

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- Define $L : \mathbf{M} \to \text{Hit}(n)$ by $L([(S,f)], q_3, \ldots, q_n) = b_S(0, q_3, \ldots, q_n)$.

The Labourie Conjecture

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Given $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(n, \mathbb{R})$ Hitchin, there exists a unique ρ-equivariant minimal surface in the symmetric space.

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- Let $\mathbf{M} \;!\; \mathbf{T}_g$ be the holomorphic vector bundle with fiber $\mathbf{M}j_{[(S;f)]} = \ {}^{n}_{i=3} H^0(S; K^i):$
- Define $L : \mathbf{M} \;!\;$ Hit(n) by $L([(S;f)]; q_3; : : : ; q_n) = b_S(0; q_3; : : : ; q_n):$
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Labourie Conjecture

Given $\ : \ {}_1(\Sigma_g) \;!\;$ PSL$(n; \mathrm{R})$ Hitchin, there exists a unique -equivariant minimal surface in the symmetric space.

- At the time, known for $n = 2; 3$ (Labourie, Loftin).
- A positive resolution of the conjecture means that L provides a MCG(Σ_g)-equivariant parametrization of Hit(n). Note also the inclusion $\mathbf{T}_g \;!\; \mathbf{M}$ is holomorphic.

In general, recall $G{=}K$ is a symmetric space, and let be a G-invariant metric.

Generalized Labourie Conjecture

Given : $_1(\Sigma_g)$! G in any higher Teichmüller space, there exists a unique minimal surface in $(G{=}K;\ )$:

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The rank of G is the dimension of a maximal flat subspace of $G{=}K$. For example: $\text{rank}(\text{PSL}(n; \text{R})) = n \quad 1:$

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Theorem (Labourie, Collier-Toulisse-Tholozan, Schoen, Wan)

The generalized Labourie conjecture holds for all Hitchin components and maximal components for Lie groups of rank 2.

Hitchin: $\mathrm{PSL}(3; \mathbb{R})$, $\mathrm{PSp}(4; \mathbb{R})$; G_2^0 (Labourie), Maximal: $\mathrm{PSL}(2; \mathbb{R})^2$ (Schoen, Wan), and Hermitian Lie groups like $\mathrm{PSO}_0(2; n)$ (Collier-Tholozan-Toulisse). See also work of Nie on G_2^0.

One simpler example of a higher Teichmüller space is $\mathbf{T}_g^n \subset \mathrm{Rep}(\mathrm{PSL}(2,\mathbb{R})^n)$.

Theorem (Marković, 2021)

For all genus g large enough and $n \geq 3$, there exists product of n discrete faithful representations $\rho = (\rho_1, \ldots, \rho_n) : \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})^n$ with multiple minimal surfaces.

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Consequently, the generalized Labourie conjecture fails for $\mathbf{T}_g^n$, $n \quad 3.$

Marković-S.-Smillie, 2022

New geometric proof of the above result, works for all genus $g \quad 2.$

Theorem (S.-Smillie, 2022)

For all $g \geq 3$ and $n \geq 4$, there exists a Hitchin representation $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(n, \mathbb{R})$ with multiple minimal surfaces. That is, Labourie's conjecture fails for $n \geq 4$.

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More generally, let G be simple split real or a product of simple split real Lie groups. We say that $\rho : \pi_1(\Sigma_g) \to G$ is Hitchin if each factor is Hitchin. Let ν be any G-invariant metric on G/K.

Theorem (S.-Smillie, 2022)

For all $g \geq 3$ and G as above with rank ≥ 3, there exists a Hitchin representation $\rho : \pi_1(\Sigma_g) \to G$ with multiple minimal surfaces in $(G/K, \nu)$.

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- The results hold for every known higher Teichmüller space such that the restriction of the Hitchin map is surjective.

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Theorem (S.-Smillie, 2022)

For all g 3 and n 4; there exists a Hitchin representation : ${}_1(\Sigma_g)$! PSL$(n;$ R$)$ with multiple minimal surfaces. That is, Labourie's conjecture fails for n 4.

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- The results hold for every known higher Teichmüller space such that the restriction of the Hitchin map is surjective.
- We can prove the same result as well for $g = 2:$ Writing up.

Corollaries

Corollary

For $n \geq 4$, $L : \mathbf{M} \to \mathrm{Hit}(n)$ has no continuous section.

Corollary

For all G as above with rank at least 3, there exists a Hitchin representation with multiple area minimizing minimal surfaces.

Questions

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What are the fibers of $L : \mathbf{M} \to \mathrm{Hit}(n)$?

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Is there a "good" parametrization of $\mathrm{Hit}(n)$?

Unstable minimal surfaces

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- We produce a Hitchin representation together with an unstable minimal surface. Since it must have a stable minimal surface as well, it must have at least two minimal surfaces.

Heuristic: minimal maps to buildings

Fix a Riemann surface S.

- Given $[\ ] \in \mathrm{Rep}(\mathrm{PSL}(n, \mathbb{C}))$ with $h_S([\ ]) = (0, q_3, \ldots, q_n)$, there exists a Hitchin ray $(\ _R)_{R \in [1, \infty)}$: a path of representations starting at with $h_S(\ _R) = (0, R^3 q_3, \ldots, R^n q_n)$ and minimal maps h_R.

Heuristic: minimal maps to buildings

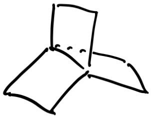
Fix a Riemann surface S.

- Given [ ] 2 Rep(PSL(n; C)) with $h_S([\ ]) = (0; q_3; :::; q_n)$, there exists a Hitchin ray $(\ _R)_{R\,2\,[1;\ 1\ )}$: a path of representations starting at with $h_S(\ _R) = (0; R^3 q_3; :::; R^n q_n)$ and minimal maps h_R.
- As $R\ !\ 1$ the minimal maps h_R "converge" to a minimal map to a building of rank $n\ \ 1$.

Rank 1 (M.Wolf):

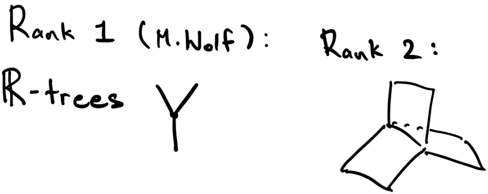
$\mathbb{R}$-trees

Rank 2:



Fix a Riemann surface S.

- Given $[\rho] \in \mathrm{Rep}(\mathrm{PSL}(n,\mathbb{C}))$ with $h_S([\rho]) = (0, q_3, \ldots, q_n)$, there exists a Hitchin ray $(\rho_R)_{R \in [1,\infty)}$: a path of representations starting at ρ with $h_S(\rho_R) = (0, R^3 q_3, \ldots, R^n q_n)$ and minimal maps h_R.
- As $R \to \infty$ the minimal maps h_R "converge" to a minimal map to a building of rank $n - 1$.



- Heuristic: minimal surfaces in rank k buildings look like minimal surfaces in $\mathbb{R}^k$. For $k \geq 3$, minimal surfaces in $\mathbb{R}^k$ are often unstable.

Minimal maps from the cameral cover

- Starting with $(0, q_3, \ldots, q_n)$ take the branched cover $: S_B \to S$ and the holomorphic $\mathfrak{t}^{\mathbb{C}}$-valued 1-form

$$= \begin{pmatrix} {}_1 & & \\ & \ddots & \\ & & {}_n \end{pmatrix}$$

with ${}_1(\ ) := \sum_j {}_j = 0$ and $q_2 = {}_2(\ ) = 0$, $q_i = {}_i(\ )$ for $i > 2$.

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- Starting with $(0, q_3, \ldots, q_n)$ take the branched cover $\pi : S_B \to S$ and the holomorphic $\mathfrak{t}^{\mathbb{C}}$-valued 1-form
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with $\sigma_1(\lambda) := \sum_j \lambda_j = 0$ and $q_2 = \sigma_2(\lambda) = 0$, $q_i = \sigma_i(\lambda)$ for $i > 2$.
- For each j, $\mathrm{Re}\,\lambda_j$ is a harmonic 1-form. The cohomology class of $\mathrm{Re}\,\lambda_j$ determines a representation $\chi_j : \pi_1(S_B) \to (\mathbb{R}, +)$.

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- Lifting each ${}_j$ to $\tilde{S}_B$ and integrating the real part, we obtain a minimal map

$$f = (f_1, \ldots, f_n) : \tilde{S}_B \to \mathbb{R}^n, \ f_j(z) = \int_p^z \mathrm{Re}\ \tilde{\ }_j$$

  that is $\ = (\ {}_1, \ldots, \ {}_n)$-equivariant and whose image is contained in the hyperplane $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_j x_j = 0\}$.

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- Identifying the hyperplane with $\mathbb{R}^{n-1}$, we write $f : \tilde{S}_B \to \mathbb{R}^{n-1}$.

Minimal maps from the cameral cover

- Taking stock, we have a ray of representations $R \mapsto \rho_R : \pi_1(S) \to G = \mathrm{PSL}(n, \mathbb{C})$ with minimal maps h_R and an equivariant minimal map $f : \tilde{S}_B \to \mathbb{R}^{n-1}$.

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- Let $B \subset S_B$ be the branch locus of $\pi : S_B \to S$. Lift h_R to $\tilde{h}_R : \tilde{S}_B \to (G/K, \ )$. The following is a consequence of an estimate of T. Mochizuki.

Loosely stated proposition

For "generic" Hitchin rays, the intrinsic data of $\tilde{h}_R$, rescaled by $1/R^2$, converges locally uniformly on $\tilde{S}_B - B$ to the intrinsic data of f. The convergence at $z \in \tilde{S}_B$ is $O(e^{-cd(z,B)R})$.

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- That is, away from B, the intrinsic data of the limiting minimal map to the building should look like f.
- **Remark:** for general G, we have a branched covering $S_B \to S$ with $\mathrm{Deck}(S_B/S)$ inside the Weyl group of the Lie algebra of G, and a minimal map to $\mathbb{R}^{\mathrm{rank} G}$.

Minimal maps from the cameral cover

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Remark: in genus 2, every equivariant minimal surface is contained in a 2-plane, and hence stable!

The index of minimal maps

Set $G = PSL(n; \mathbb{C})$; $K = PSU(n; \mathbb{C})$:

- Formally, variations of a minimal map $h : \tilde S \to (G/K;\ )$ are sections $\dot h \in \ _S(h\ TG/K)$: We write "$h_t = h + t\dot h$:"

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- The index of h, $\mathrm{Ind}(h)$, is the maximal dimension of a subspace of $\Gamma_S(h^* TG/K)$ on which

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- For $\pi : S_B \to S$ and $f : \tilde{S}_B \to \mathbb{R}^{n-1}$, $\text{Ind}_\pi(f)$ is the maximal dimension of a subspace of $\Gamma_{S_B}(f^*T\mathbb{R}^{n-1})$ consisting of Deck(S_B/S)-invariant variations on which
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Theorem (S.-Smillie 2022)

$\liminf_{R\to\infty} \ \mathrm{Ind}(h_R) \quad \mathrm{Ind}(f).$

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Using the Hitchin section, we obtain a path of Hitchin representations $\rho_R : \pi_1(\Sigma_g) \to PSL(n,\mathbb{R})$ with minimal maps $h_R : \tilde{S} \to PSL(n,\mathbb{R})/PSO(n,\mathbb{R})$ such that

$$\liminf_{R\to\infty} \mathrm{Ind}(h_R) \geq \mathrm{Ind}(f).$$

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For R sufficiently large, h_R is unstable, and hence ρ_R admits at least two minimal surfaces.

A Question

For $\mathrm{PSL}(2,\mathbb{R})^n$, we can show $\mathrm{Ind}(h_R)$ is non-decreasing with R and $\lim_{R\to\infty} \mathrm{Ind}(h_R) = \mathrm{Ind}(f)$ (Marković-S.-Smillie).

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For more general G, we have Hitchin rays such that $\text{Ind}(h_R) > \text{Ind}(f)$, although we have no example in the Hitchin component.

Question

What more can we say about $\text{Ind}(h_R)$ in terms of $\text{Ind}(f)$?

Thanks for listening