Degenerating families of affine spheres and minimal maps to buildings

Michael Wolf, Georgia Institute of Technology

Joint Teichmuller Theory Seminar, December 2022

Overview, I

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Joint work with John Loftin and Andrea Tamburelli.

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In the end, if we rescale $\mathbb{H}^2$ by $s^{\frac{1}{2}}$, the sequence of harmonic maps converges to a harmonic map to a real tree whose geometry reflects that *horizontal* direction L of q_0.

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Asymptotic holonomy given by invariants of quad diff. $\longrightarrow$ Asymptotic holonomy given by invariants of cubic diff.

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Thus $[\gamma]$ may be represented by a unique geodesic in this metric. which is a collection of straight lines in the Euclidean portions, which turn through some angle $\geqslant \pi$ at the cone points. Write $\gamma = \gamma_n \cup \gamma_{n-1} \cup ... \cup \gamma_1$.

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where we integrate the cosine of the angle with the relevant foliation.

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Theorem

[Loftin-Tamburelli-W]

$$\lim_{s \to +\infty} \frac{\log \| \operatorname{hol}(\rho_s)(\gamma) \|}{s^{\frac{1}{3}}} = \sum_{j=1}^{n} \nu_j. \tag{3}$$

so the two constructions agree: asymptotic holonomy is given by local holomorphic data.

Asymptotic Holonomy discussion

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[Loftin-Tamburelli-W]

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And we are saying that the total holonomy is just given by adding up the dominant eigenvalues of the flat singular holonomy... as if we could just lay those segments end-to-end and multiply holonomies. We'll describe the reasons for this a bit later.

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Theorem

[Loftin-Tamburelli-W]

$$\lim_{s \to +\infty} \frac{\log \| \mathrm{hol}(\rho_s)(\gamma) \|}{s^{\frac{1}{3}}} = \sum_{j=1}^{n} \nu_j. \tag{4}$$

This theorem is technically a bit strange. After all, the zeroes – always important in holomorphic considerations – do not seem to play a role.

And we are saying that the total holonomy is just given by adding up the dominant eigenvalues of the flat singular holonomy... as if we could just lay those segments end-to-end and multiply holonomies. We'll describe the reasons for this a bit later.

This statement has a somewhat "tropical" aspect to it that I do not understand.

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We prove some results that reflect the structure of this limiting harmonic map.

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Moreover, we show

Theorem

[Loftin-Tamburelli-W] Given a representation $\rho : \pi_1(S) \to \mathrm{Isom}(B)$, if $\rho(\pi_1(S))$ does not preserve any totally geodesic flat subspace in B, then there is a unique equivariant conformal harmonic map $h_\rho : \widetilde{S} \to B$.

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This results suggests that with suitable hypotheses, all the buildings are the same, and (perhaps) all come from the holomorphic data.

As another side remark, we do note that for (many) triangle groups, the Hitchin component is a disk bounded by a circle, which is satisfying.

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