The geometry of complementary subsurfaces to simple closed hyperbolic multigeodesics

joint w/ Aaron Calderon

Board 1;

Main question;

Are simple (closed) geodesics on a hyperbolic surface X biased?

Primitive (closed) geodesics;

- completeness $\sim$ complete geodesic through every tangent vector
- $p(X,L) = \#\{\text{primitive closed geodesics on } X \text{ of length} \leq L\}$
  $\sim e^L/L$ (Selberg, Huber '56, Margulis '70)
- $\mu_L = \frac{1}{p(X,L)} \sum_{\alpha \text{ primitive}} \alpha_* \mathrm{Leb} \xrightarrow{*} \mu_{\mathrm{Liouville}}$ (Bowen '80)
- Long random geodesics tessellate X like a Poisson line process (Athreya, Lalley, Sapir, Wroten '17)

Board 2;

Simple (closed) geodesics;

- $\dim_{\mathcal{H}}\left(\bigcup_{\gamma \text{ simple}} \gamma\right) = 1$ (Birman Series '85)
- $s(X,L) = \#\{\text{simple closed geodesics on } X \text{ of length} \leq L\}$
  $\sim B(X)\, L^{6g-6}$ (Mirzakhani '08)
- $\mu_L := \frac{1}{s(X,L)} \sum_{\gamma \text{ simple}} \gamma_* \mathrm{Leb} \to \mu \perp \mu_{\mathrm{Liouville}}$ (Erlandsson Souto '21)

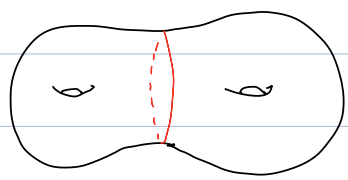
Board 3;

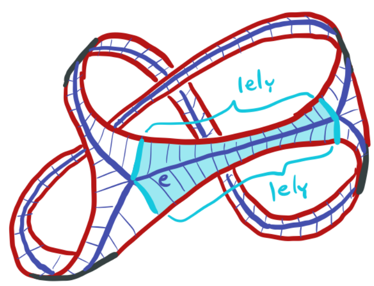
Question;

How to encode the geometry of complementary surfaces to simple closed geodesics?

Answer;

Using metric ribbon graphs





Board 4;

Main theorem (AH, Calderon '22)

As lengths go to infinity, complementary subsurfaces to simple closed hyperbolic geodesics equidistribute to the Kontsevich measure on the corresponding moduli space of metric ribbon graphs.

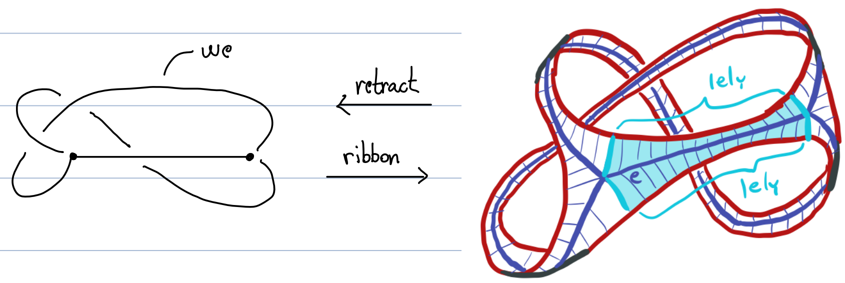
Answer to the original question;

Simple closed geodesics on hyperbolic surfaces are as unbiased as they could be

Board 5:

Ribbon graph:

A graph with a cyclic ordering of the edges incident to every vertex



What do they encode topologically?

$\{$ribbon graphs$\}$ $\longleftrightarrow$ $\{$deformation retractions of surfaces with boundary$\}$

$\sim$ genus $\sim$ # boundary components

Board 6:

Metric ribbon graphs:

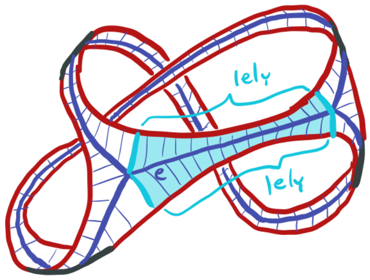
A ribbon graph with an assignment of length to each edge

What they encode geometrically?

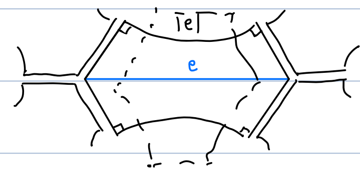
$\{$metric ribbon graphs of genus g with b boundary comp$\}$ $= MRG_{g,b}$ $\longleftrightarrow$ $\{$hyperbolic surfaces of genus g with b totally geodesic boundary components$\}$ $:= \mathcal{M}_{g,b}$

Board 7:

Spine:



Reconstruct:



Theorem (Luo):

The spine map

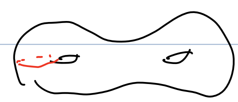
$S : \mathcal{M}_{g,b} \to MRG_{g,b}$

is a homeomorphism that 'preserves' boundary length.

Board 8:

For simplicity:

Restrict to non-separating simple closed geodesics



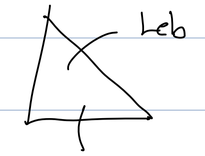
Given $X \in \mathcal{M}_g$ closed hyperbolic surface of genus g

α simple non-separating closed geodesic

$RSC_\alpha(X) \in MRG_{g-1,2}(1,1)$

↳ rescale ∘ spine ∘ cut

Board 9;

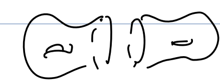


Kontsevich measure;

μ_{Kon} = Lebesgue measure on $MRG_{g-1,2}(1,1)$

Counting;

$sns(X,L) = \#\{$simple non-separating geodesics on X of length $\leq L\}$



Main theorem (AH, Calderon '22)

For <u>every</u> $X \in \mathcal{M}_g$

$$\underbrace{\lim_{L\to\infty} \frac{1}{sns(X,L)} \sum_{\substack{\alpha \text{ sns} \\ \ell_\alpha(X)\leq L}} \delta_{RSC_\alpha(X)}}_{\eta_{X,L}} = \mu_{Kon}$$

on $MRG_{g-1,2}(1,1)$

$MRG_{1,1}^{(1)} \times MRG_{1,1}^{(1)}$

$\mu_{Kon} \times \mu_{Kon}$

Board 10;

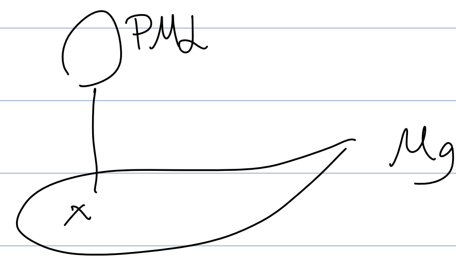
Pipeline of proof;

$\begin{bmatrix}\text{equidistribution} \\ \text{on } MRG_{g-1,2}(1,1)\end{bmatrix} \xrightarrow{\text{integration}} \begin{bmatrix}\text{counting in} \\ MRG_{g-1,2}(1,1)\end{bmatrix} \xrightarrow{RSC} \begin{bmatrix}\text{counting parametrized} \\ \text{by } X \in \mathcal{M}_g\end{bmatrix}$

$\xrightarrow[\text{and unfolding}]{\text{averaging}} \begin{bmatrix}\text{equidistribution} \\ \text{in } \mathcal{M}_g\end{bmatrix} \xrightarrow[\text{earthquake}]{\text{ergodicity}} \begin{bmatrix}\text{total mass} \\ \text{computation}\end{bmatrix} \xrightarrow[\text{Do, Mondello}]{wp \to kon} [\text{End}].$

$P^1\mathcal{M}_g = \{$ bundle of unit length measured geodesic laminations over $\mathcal{M}_g$ $\}$

Board 11;

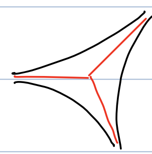


(X, λ)

in which direction to twist

Board 12;

Board 13;



Board 14;

Board 15;

Board 16;